# Solving the KdV Equation Using the PINN Method

Rumin Zhou *

College of Science, Xizang University, Lhasa, 850000, China

* Corresponding Author Email: 2251472680@qq.com

**Abstract.** Traditional numerical methods such as the Finite Difference Method (FDM) often exhibit significant numerical dissipation and dispersion errors when solving the Korteweg-de Vries (KdV) equation, particularly in long-term simulations or complex wave interactions like double soliton collisions. To address these limitations, this study proposes an improved approach based on Physics-Informed Neural Networks (PINNs). By embedding the physical constraints of the KdV equation directly into the loss function, and employing a lightweight fully-connected neural network with four hidden layers and automatic differentiation, the method enhances both accuracy and generalization capability. Numerical experiments on single and double soliton cases demonstrate that the PINN method significantly outperforms FDM and conventional Artificial Neural Networks (ANNs). For instance, in the single soliton case, the maximum error of PINN is reduced to 8.36e-03, with an L2 error of 1.33e-02, while FDM results in larger errors due to numerical artifacts. In the double soliton collision scenario, PINN effectively captures the nonlinear interaction process with minimal error accumulation. Although slight discrepancies are observed near the wave crest-especially for the faster wave-the overall waveform and dynamic behavior are accurately preserved. The absolute error distribution further confirms the stability and precision of the PINN solution across the computational domain. These results underscore the critical role of physical constraints in improving the modeling of nonlinear wave phenomena. The proposed PINN framework offers a robust, efficient, and high-fidelity alternative for solving complex PDEs in scientific and engineering applications.

**Keywords:** Physics-Informed Neural Networks, Korteweg-de Vries Equation, Soliton Wave.

## 1. Introduction

In 1895, Dutch scientists Korteweg and de Vries jointly proposed the KdV equation [1], which aims to theoretically explain the solitary wave phenomenon observed by Scott Russell [2]. As a nonlinear partial differential equation,

Traditional numerical methods, such as the Finite Difference Method (FDM), have been widely applied to solve the KdV equation. For instance, Bi Yanming et al. utilized FDM for a variable-coefficient equation, confirming scheme convergence but also highlighting limitations like simplistic handling of boundary conditions [3]. While FDM is conceptually straightforward, it often suffers from numerical dissipation and dispersion errors, which can significantly degrade solution accuracy, especially for long-term simulations or complex interactions like double soliton collisions. Alternatively, data-driven approaches like Artificial Neural Networks (ANNs) have shown promise; Yuan Dongfang et al. demonstrated their capability in solving elliptic equations with high accuracy [4]. However, pure ANN models often require large amounts of training data and may lack generalizability and robustness for complex physical problems, as they do not inherently respect the underlying physical laws.

This study addresses these limitations by proposing a novel application of Physics-Informed Neural Networks (PINNs) to solve the KdV equation. The core innovation of PINN lies in its seamless integration of deep learning with physical governing equations [5]. By embedding the KdV equation directly into the loss function of a neural network, the PINN method ensures that the solution not only fits initial and boundary data but also strictly adheres to the physical constraints described by the PDE. This approach represents a paradigm shift from purely data-driven models to physics-constrained learning. A key contribution of this work is the design of a lightweight, fully-connected 4-layer network with Tanh activation, which reduces computational complexity while maintaining high accuracy and fast convergence. This efficient architecture makes the method particularly suitable for practical engineering applications involving large-scale nonlinear problems [6]. Through comparative numerical experiments on single and double solitary wave cases, this research rigorously verifies that the PINN method significantly outperforms both traditional FDM and conventional ANN approaches in accuracy and generalization, underscoring the critical role of physical priors in scientific machine learning.

## 2. Method Introduction

The Finite Difference Method (FDM) is a classical numerical technique for approximating solutions to differential equations [7]. Its core principle involves discretizing continuous mathematical problems by replacing derivatives with finite differences. The continuous domain (e.g., space and time) is overlayed with a grid, and derivatives at a point are approximated using the function values at neighboring grid points. While FDM is conceptually straightforward and widely implemented, it suffers from significant drawbacks when applied to nonlinear wave equations like the KdV equation. These include numerical dissipation (loss of wave energy) and dispersion (waves of different frequencies propagating at incorrect speeds), which can lead to substantial error accumulation over time, especially in long-term simulations or complex interactions like soliton collisions. An Artificial Neural Network (ANN) is a computational model inspired by biological nervous systems. It consists of interconnected layers of artificial neurons that learn complex mappings between input and output data through weighted connections. ANNs are renowned as universal function approximators [8-10], meaning they can theoretically represent any continuous function given sufficient network complexity. In the context of solving PDEs, an ANN can be trained to map spatial and temporal coordinates (x, t) directly to the solution u (x, t). However, a major limitation of a purely data-driven ANN is its reliance on large volumes of high-fidelity training data (e.g., from expensive experiments or precise numerical simulations). Furthermore, without built-in physical constraints, the ANN's predictions may violate the fundamental physical laws governing the system, leading to poor generalization outside the training data range.

The Physics-Informed Neural Network (PINN) method represents a paradigm shift by cleverly integrating deep learning with physical prior knowledge [8]. Its superiority in solving problems like the KdV equation stems from three core ideas. In this study, we implement a lightweight yet powerful PINN architecture specifically for the KdV equation. The network is a fully connected neural network with only 4 hidden layers, using the Tanh activation function. This design choice reduces the number of parameters and computational complexity compared to deeper networks, while maintaining excellent approximation capabilities. The network intelligently adjusts its parameters during training to minimize the combined loss, ensuring the output solution is not just a data fit but a physically admissible one. This approach demonstrates unique advantages for the KdV equation [9], effectively capturing nonlinear wave dynamics with high accuracy and robust generalization.

## 3. Single Solitary Wave Case

### 3.1. Numerical Comparison

The KdV equation has solitary wave solutions and can accurately characterize the nonlinear effects and dispersion effects of waves during propagation. Its standard form is expressed as:

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

Initial Condition Setting:

$$u(x, 0) = 2\mathrm{sech}^2(x) \tag{2}$$

Analytical Solution:

$$u(x,t) = A\mathrm{sech}^2\left(\sqrt{\frac{A}{2}}(x - ct - x_0)\right) \tag{3}$$

The analytical solution of the single solitary wave accurately describes the shape of the solitary wave. Parameters A, c, and $x_0$ determine the amplitude, speed, and initial position, respectively, where $-20 \leq x \leq 20$, $0 \leq t \leq 2.0$, $c = 1.0$, and $x_0 = 0.0$. The initial condition represents a solitary wave crest at $t = 0$, which conforms to the characteristics of the solitary wave solution of the KdV equation and can simulate the shape of the solitary wave at the initial moment.

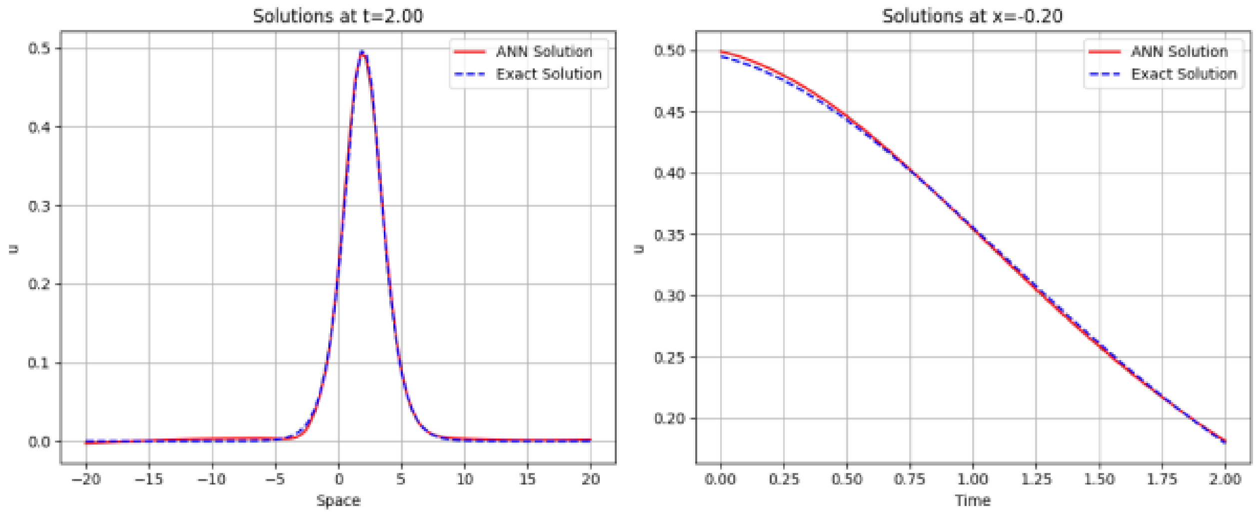


**Figure 1.** Comparison between ann predicted solution and benchmark solution at t=2.0 and x=0.2

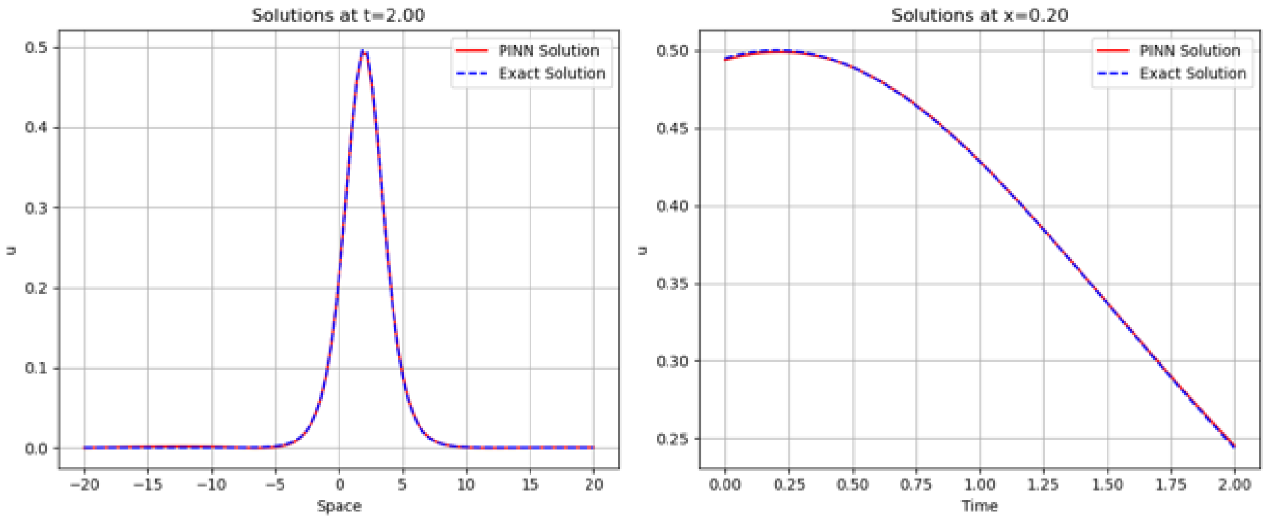


**Figure 2.** Comparison between pinn predicted solution and benchmark solution at $t = 0.2$ and $x = 0.2$

From Figures 1 and 2, the overall shape and amplitude of the two curves (PINN predicted solution and benchmark solution) are highly consistent. However, the maximum error of FDM is 8.36e-03 and the L2 error is 1.33e-02, with errors mainly concentrated near the solitary wave crest and small errors in other regions. In contrast, the curves of the PINN predicted solution and the benchmark solution show the same trend over time, and the predicted values remain very close to the actual values.

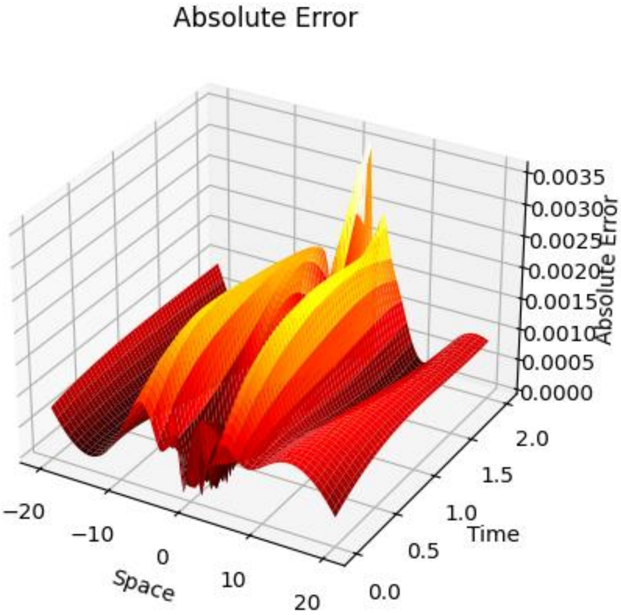


**Figure 3.** Absolute error heatmap of pinn for solving the kdv equation (single)

Deng Zhenguo and Ma Heping pointed out that the errors in Figure 3 mainly come from three aspects: truncation error (error caused by discretization of continuous problems), rounding error (error caused by the limitation of computer floating-point operation accuracy), and discretization error (error caused by discrete approximation itself). In particular, discretization error is particularly significant when dealing with high-order derivative terms, which may lead to a decrease in the accuracy of numerical solutions. The literature also mentions that these errors can be effectively reduced by reasonably selecting collocation points and polynomial orders, thereby improving the accuracy of numerical solutions. Solitary waves are highly sensitive to small perturbations in initial amplitude or phase, and this sensitivity mainly originates from nonlinear effects. Nonlinear effects cause errors to grow exponentially with time, resulting in a decline in the accuracy of numerical solutions [11].

### 3.2. Error Comparison

All three methods can solve single solitary waves relatively accurately, but there are differences in accuracy. FDM has large errors due to numerical dissipation and dispersion; although ANN has advantages in global fitting, the error in the peak region is relatively high; PINN performs the best with the lowest errors in all three indicators, balancing global and local accuracy, which reflects the advantage of combining physical constraints with neural networks.

## 4. Double Solitary Wave Case

### 4.1. Numerical Comparison

The initial condition is set as $u(x, 0) = 2\mathrm{sech}^2(x + 5) + 2\mathrm{sech}^2(x - 5)$, which indicates that two solitary waves are located at $x + 5$ and $x - 5$ respectively. This condition can simulate the initial state of double solitary waves before collision. To cope with the complexity of the collision process, a dynamic loss weight adjustment strategy is adopted to intelligently adjust the weights of various losses according to the training progress and balance the training focus. This method can more effectively capture the subtle changes during collision, improve the fitting ability and convergence stability of the model, thereby enhancing the accuracy and reliability of double solitary wave collision simulation.

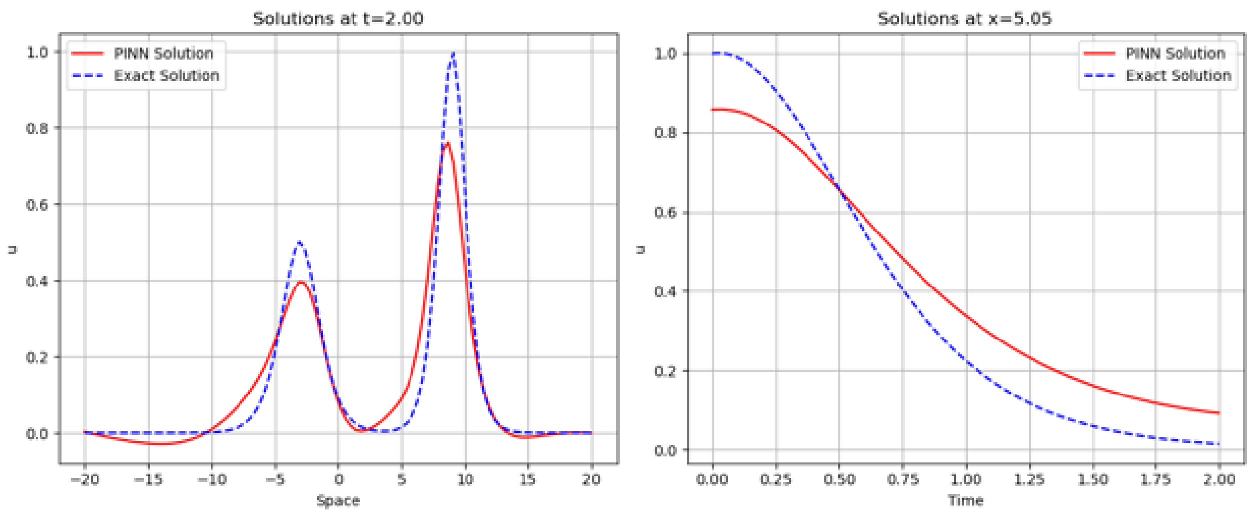


**Figure 4.** Comparison between pinn predicted solution and benchmark solution for double solitons at $t = 2.0$ and $x = 5.05$

As can be seen from Figure 4, there is a slight discrepancy in the amplitude and position near the wave crest of the PINN solution; PINN is more accurate in predicting the slow wave (left low peak) but slightly underestimates the fast wave (right high peak) at the wave top, while the overall trend and shape are good.

### 4.2. Error Comparison of Double Solitary Waves

The differences between the three methods in solving double solitary waves are more obvious. FDM has the largest errors, showing significant accumulation of numerical dissipation and dispersion; although ANN has been improved, its accuracy is still limited; PINN has the lowest errors, significantly outperforming the previous two methods. It can effectively suppress the error accumulation caused by nonlinear interference, demonstrating advantages in complex wave problems.

## 5. Conclusion

This study successfully demonstrates the superior efficacy of the Physics-Informed Neural Network (PINN) method in solving the Korteweg-de Vries (KdV) equation, effectively overcoming the inherent limitations of accuracy and efficiency associated with the traditional Finite Difference Method (FDM). The numerical experiments conducted on both single and double solitary wave scenarios provide compelling evidence. In the single soliton case, the PINN solution exhibits remarkable consistency with the analytical benchmark, accurately capturing the wave's amplitude and propagation dynamics.

The absolute error heatmap further confirms that errors are minimal and well-controlled across the spatial and temporal domain.

The superiority of PINN becomes even more pronounced in the more complex double soliton collision test. The method effectively simulates the non-trivial interaction process where the two solitary waves pass through each other and emerge unchanged. While there is a slight discrepancy in the amplitude of the faster wave, the overall trend and shape prediction are excellent.

The adoption of a dynamic loss weight adjustment strategy was crucial here, intelligently balancing the training focus to capture subtle changes during the collision and enhance convergence stability. The error comparison conclusively shows that PINN significantly suppresses the error accumulation caused by nonlinear interference, a common pitfall for FDM and ANN.

In summary, the integration of physical constraints into the neural network's learning process is the key factor behind PINN's high accuracy and robust generalization ability. This research validates PINN as a powerful and efficient tool for tackling nonlinear wave problems. Looking forward, while the current random sampling strategy is effective, further improvements can be achieved by implementing an adaptive sampling strategy. Such a strategy would intelligently increase the sampling density in regions with large errors and near boundaries, thereby more accurately characterizing boundary effects and improve the model's overall accuracy and stability for even more challenging problems. This work paves the way for the application of lightweight, physics-aware neural networks in broader scientific and engineering computations.

## References

[1] Qiu Tianwei, Wei Guangmei, Song Yuxin, et al. Research on new solitary wave solutions of KdV-type equations based on the PINN method [J]. Applied Mathematics and Mechanics, 2025, 46 (01): 105 - 113.

[2] Wang Lin, Liu Ziqiang, Ouyang Zigen, et al. Orbital stability of solitary wave solutions for the combined KdV equation [J]. Journal of University of South China (Natural Science Edition), 2023, 37 (05): 87 - 91.

[3] Bi Yanming, Yin Zhe. Finite difference method for the variable-coefficient Cattaneo equation [J]. Journal of Shandong Normal University (Natural Science Edition), 2020, 35 (04): 436 - 439.

[4] Yuan Dongfang, Liu Wenhui, Cui Guimei, et al. Numerical solution of elliptic differential equations based on artificial neural networks [J]. Journal of Ningxia University (Natural Science Edition), 2022, 43 (01): 6 - 11.

[5] Wei Chang, Fan Yuchen, Zhou Yongqing, et al. Physics-informed neural networks with time weights for solving unsteady partial differential equations [J]. Acta Mechanica Sinica, 2025, 57 (03): 755 - 766.

[6] Chen Wankun, Yuan Chunxin. Application of the variable-coefficient KdV equation in ocean internal solitary waves [J]. Periodical of Ocean University of China (Natural Science Edition), 2020, 50 (08): 19 - 24.

[7] Huang Guannan, Wang Jingyue, Wang Meiqing. Neural network solver for nonlinear partial differential equations based on the improved Euler method [J]. Journal of Fuzhou University (Natural Science Edition), 2024, 52 (04): 396 - 403.

[8] Huang Ruilong, Kuang Yuanyuan, ULLAH Arif, et al. Analysis of solving the nonlinear Schrödinger equation by physics-informed neural networks combined with optimization algorithms [J/OL]. Chinese Journal of Quantum Electronics, 1 - 15 [2025 – 09 - 25].

[9] Chen Haolong, Tang Xinyue, Wang Runhua, et al. Research on multi-medium nonlinear transient heat conduction problems based on physics-informed neural networks [J]. Acta Mechanica Sinica, 2025, 57 (01): 89 - 102.

[10] K.A. LUONG, M. A. WAHAB, J. H. LEE. Simultaneous imposition of initial and boundary conditions via decoupled physics-informed neural networks for solving initial-boundary value problems [J]. Applied Mathematics and Mechanics (English Edition), 2025, 46 (04): 763 - 780.

[11] Deng Zhenguo, Ma Heping. Optimal error estimates for Fourier spectral approximations of the generalized KdV equation [J]. Applied Mathematics and Mechanics, 2009, 30 (1): 30 - 39.